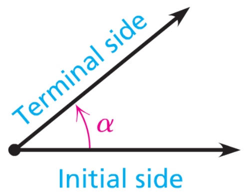
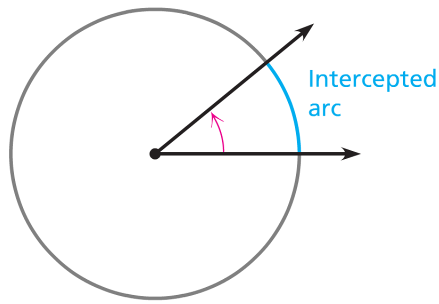
Angles and Degree Measure

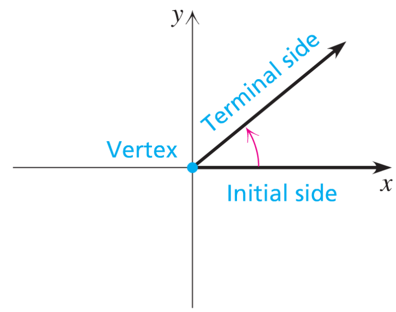
An **angle** can be formed by rotating one ray away from a fixed ray indicated by an arrow. The fixed ray is the **initial side** and the rotated ray is the **terminal side**. An angle whose vertex is the center of a circle is a **central angle**, and the arc of the circle through which the terminal side moves is the **intercepted arc**. An angle in **standard position** is located in a rectangular coordinate system with the vertex at the origin and the initial side on the positive x-axis.



Angle α



Central angle



Angle in standard position

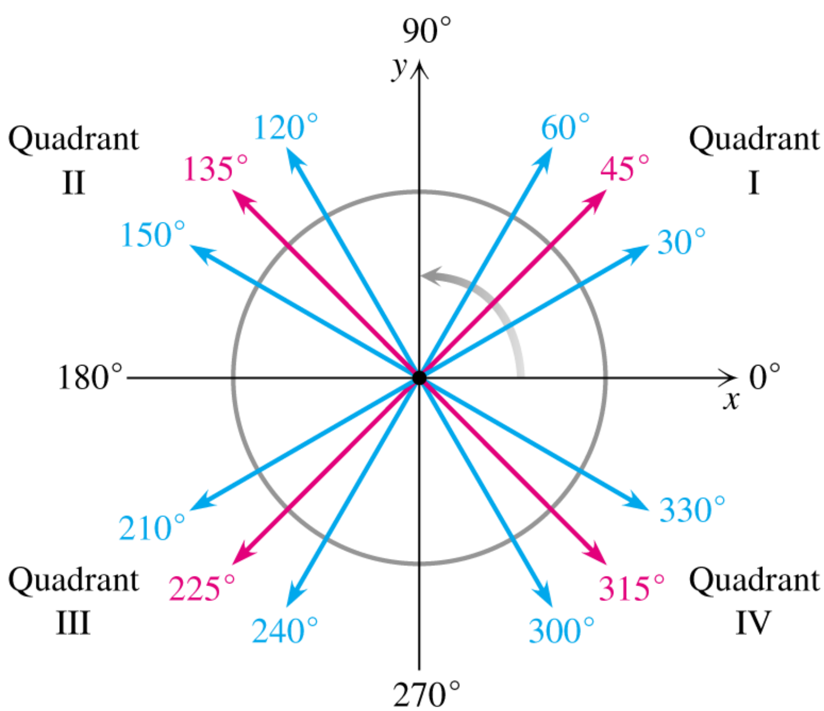
Degree Measure of Angles

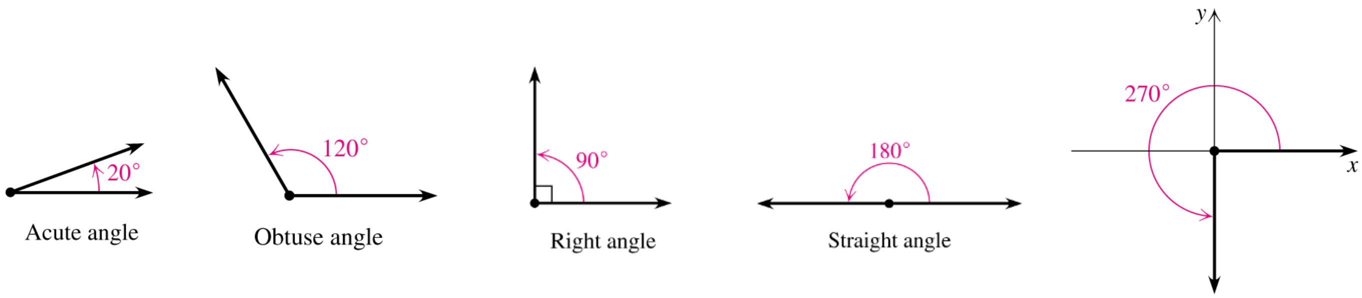
The measure, $m(\alpha)$, of an angle α is the amount of rotation from the initial side to the terminal side, and is found by using any circle centered at the vertex. An angle that forms a complete circle arc is 360°.

The **degree measure of an angle** is the number of degrees in the intercepted arc of a circle centered at the vertex.

Counterclockwise rotation—positive angle **Clockwise rotation—negative angle**

An angle in standard position is said to lie in the quadrant where its terminal side lies.





Acute angle Obtuse angle Right angle Straight angle Quadrantal angle

Acute angle—An angle with a measure between 0° and 90°.
Obtuse angle—An angle with a measure between 90° and 180°.
Straight angle—An angle with a measure of exactly 180°.
Right angle—An angle with a measure of exactly 90°.
Quadrantal angle—An angle in standard position whose terminal side is on an axis.

The terminal side of an angle may be rotated in either a positive or negative direction to get to its final position. It can also be rotated for more than one revolution in either direction. Two angles with the same terminal side are called **coterminal angles.**

Coterminal Angles—Angles α and β are coterminal if and only if there is an integer k such that $m(\beta) = m(\alpha) + k360°$. To find coterminal angles in degrees, add and subtract multiples of 360°.

Examples: Find two positive angles and two negative angles that are coterminal with each angle:

a) 23°

b) –146°

Examples: Determine whether the angles in each pair are coterminal:

a) –128° and 592°

b) 8° and –368°

Example: Draw each angle in standard position, then name the quadrant in which the terminal side lies.

a) 255°

b) –650°

c) 1360°

Minutes and Seconds

Historically, angles were measured by using the **degrees-minutes-seconds (DMS) format**, but with calculators it is convenient to have some fractional parts of degrees written in decimal form. Each degree is divided into 60 equal parts called **minutes** (n'), and each minute is divided into 60 equal parts called **seconds** (n'').

1 degree = 60 minutes   1 minute = 60 seconds   1 degree = 3600 seconds

Examples: Convert each angle to decimal degrees. When necessary, round to four decimal places.

a) $18°24'$ b) $-10°15'42''$ c) $27°10'20''$

Examples: Convert each angle to degree-minutes-seconds format. Round to the nearest whole number of seconds.

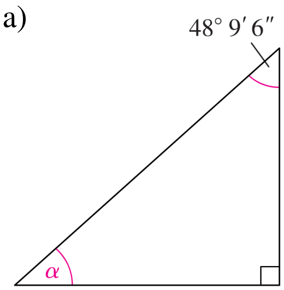
a) $56.6°$ b) $-17.45°$ c) $28.348°$

Examples: Perform the indicated operations.

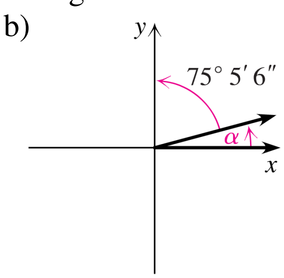
a) $15°56'45'' + 18°12'33''$ b) $36°5' - 22°33'12''$ c) $\dfrac{15°56'45''}{2}$

Examples: Find the degree measure of angle α in each figure.

a)



b)



Radian Measure, Arc Length, and Area

Unit circle: A circle of radius one that is centered at the origin.

What is the circumference of the unit circle? (Circumference of a circle: $C = \pi d = 2\pi r$)

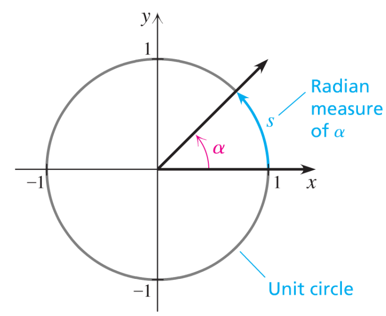
What is the arc length intercepted by a 180° angle (1/2 of the circle)?

What is the arc length intercepted by a 120° angle (1/3 of the circle)?

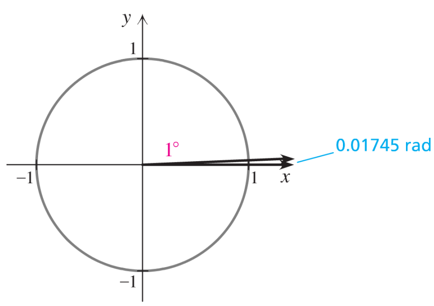
What is the arc length intercepted by a 30° angle? An angle of 225°? An angle of 210°?

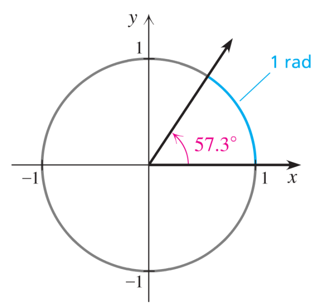
You have just calculated the radian measure of each of these angles.

The **radian measure** of the angle α in standard position is the directed length of the intercepted arc on the unit circle. **Directed length** means that the radian measure is positive if the rotation of the terminal side is counterclockwise and negative if the rotation is clockwise.



One radian: The angle that intercepts an arc with length equal to the radius of a circle. (On the unit circle, one radian is the angle that intercepts an arc of length one.)





Since the radius of the unit circle is the real number 1, without any dimension (ft., meters, etc.), the arc length, and hence the radian measure of an angle, is also a real number without any dimension. Thus, one radian (1 rad) is the real number 1.

Converting Between Radians and Degrees:

Since there are 2π radians in a circle (the circumference of the unit circle is 2π) and 360° in a circle,

$$2\pi \text{ radians} = 360°, \text{ or } \pi \text{ radians} = 180°.$$

Degrees→Radians: multiply by $\dfrac{\pi \text{ rad}}{180°}$. **Radians→Degrees:** multiply by $\dfrac{180°}{\pi \text{ rad}}$.

Examples:

Convert the degree measures to radians:

a) 210°   b) –27.2°

Convert the radian measures to degrees:

a) $\frac{5\pi}{3}$ b) 16.7 radians

Examples: Draw each angle in standard position and determine the quadrant in which each angle lies.

a) $-\frac{5\pi}{4}$ b) $\frac{10\pi}{3}$ c) 13.8   d) -2.5

Coterminal Angles:

To find coterminal angles in radians, add or subtract multiples of 2π. Make sure to find a common denominator.

Examples: Find two positive angles and two negative angles that are coterminal with the given angle.

a) $\frac{5\pi}{6}$ b) $-\frac{\pi}{4}$ c) $\frac{7\pi}{3}$ d) 1.4

Arc Length

Often, we want to find the arc length (s) on a circle of radius r, intercepted by an angle α. We can do this by determining what fraction of the circle we are looking at, then multiplying by the circumference of the circle.

Degrees: $s = \frac{\alpha}{360°} \cdot 2\pi r$

Radians: $s = \frac{\alpha}{2\pi} \cdot 2\pi r$ or $\boldsymbol{s = \alpha r}$ **This formula only works if $\boldsymbol{\alpha}$ is in radians!**

Examples:

A central angle of $\pi/2$ intercepts an arc on the surface of the earth that runs from the equator to the North Pole. Using 3950 miles as the radius of the earth, find the length of the intercepted arc to the nearest mile.

A wagon wheel has a diameter of 28 inches and an angle of 30° between the spokes. What is the length of the arc *s* (to the nearest hundredth of an inch) between two adjacent spokes?

Area of a Sector of a Circle

Finding the area (A) of a sector of a circle of radius r with central angle α, is similar to finding the arc length: Determine what fraction of the circle the sector makes up, then multiply by the area of the circle.

Degrees: $A = \dfrac{\alpha}{360°} \cdot \pi r^2$

Radians: $A = \dfrac{\alpha}{2\pi} \cdot \pi r^2$ or $\boldsymbol{A = \dfrac{\alpha r^2}{2}}$ **This formula only works if $\boldsymbol{\alpha}$ is in radians!**

Examples:

Which is bigger: a slice of pizza from a 10″ diameter pizza cut into 6 slices, or a slice from a 12″ diameter pizza cut into 8 slices?

A center-pivot irrigation system is used to water a circular field with radius 200 feet. In three hours the system waters a sector with a central angle of $\pi/8$. What area (in square feet) is watered in that time?

Angular and Linear Velocity

Velocity: The rate at which the location of an object is changing with respect to time.

Angular Velocity: The rate at which the central angle is changing for an object moving in a circle. If a point is in motion on a circle through an angle of α radians in time t, then its angular velocity ω is given by $\omega = \frac{\alpha}{t}$. Angular velocity is usually expressed as radians per unit of time (radians/hr, radians/min, radians/sec, etc.)

Examples:

Convert 650 rpm (revolutions per minute) to radians per minute.
(Use the fact that $1 \text{ revolution} = 2\pi \text{ radians}$).

Convert the angular velocity of 1600 rad/hr to rev/hr.

A 24-inch lawnmower blade rotates at a rate of 2000 rpm. What is the angular velocity in radians per second of a point on the tip of the blade?

A particle is moving in a circular path with a radius of 9 ft. at 30 radians per minutes. How fast is the particle rotating in revolutions per second?

Linear Velocity: The rate at which the position of the object is changing with respect to time. If a point is in motion on a circle of radius r through an angle of α radians in time t, then its linear velocity v is given by $v = \frac{s}{t}$, where s is the arc length determined by $s = \alpha r$.

Examples:
A propeller with a radius of 1.6 meters is rotating at 1500 revolutions per minute. What is the linear velocity in meters per second for a point on the tip of the propeller?

Find the angular velocity in radians per second for a particle that is moving along a circle with diameter 15 meters at a linear velocity of 20 meters per second.

What is the linear velocity in miles per hour of the tip of a 20-inch lawnmower blade that is rotating at 3000 rpms?

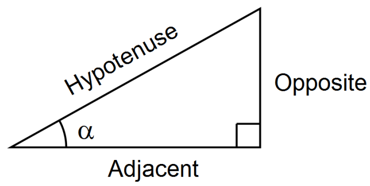
Find the angular velocity in radians per minute for a particle that is moving in a circular path at 95 mph on a circle with a radius of 8 inches.

Linear Velocity in Terms of Angular Velocity: If v is the linear velocity of a point on a circle of radius r, and ω is its angular velocity, then $v = r\omega$.

Example:
Any point on the surface of the earth (except at the poles) makes one revolution (2π radians) about the axis of the earth in 24 hours. So the angular velocity of a point on the earth is $2\pi/24$ or $\pi/12$ radians per hour. The linear velocity of a point on the surface of the earth depends on its distance from the axis of the earth. What is the linear velocity in miles per hour of a point on the equator? (Use 3950 miles as the radius of the earth).

The Trigonometric Functions

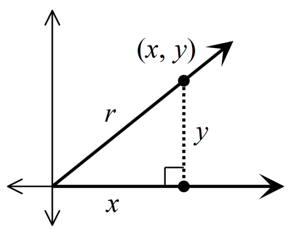
The six trigonometric functions are the sine (sin), cosine (cos), tangent (tan), cosecant (csc), secant (sec), and cotangent (cot) functions. There are several ways to define these functions of trigonometry. One of the most common mnemonic devices is SOH-CAH-TOA.



$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\alpha = \frac{\text{opposite}}{\text{adjacent}}$$

All of these ratios can be written in terms of an angle in the coordinate plane.

If (x, y) is any point other than the origin on the terminal side of an angle α in standard position and $r = \sqrt{x^2 + y^2}$, then



$$\sin\alpha = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \qquad \cos\alpha = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} \qquad \tan\alpha = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

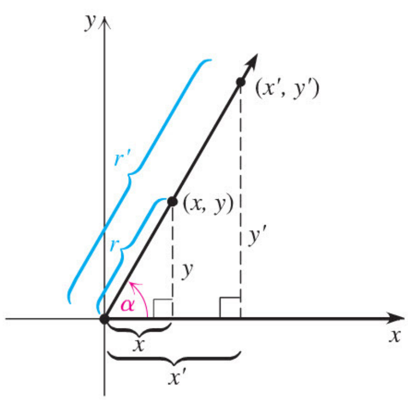
$$\csc\alpha = \frac{\text{hyp}}{\text{opp}} = \frac{r}{y} \qquad \sec\alpha = \frac{\text{hyp}}{\text{adj}} = \frac{r}{x} \qquad \cot\alpha = \frac{\text{adj}}{\text{opp}} = \frac{x}{y}$$

Reciprocal Identities: $\csc\alpha = \dfrac{1}{\sin\alpha}$ $\quad \sec\alpha = \dfrac{1}{\cos\alpha}$ $\quad \cot\alpha = \dfrac{1}{\tan\alpha}$

Examples:

Find the values of the six trigonometric functions of the angle α in standard position whose terminal side passes through (-2, -4).

To find the values of the trigonometric functions of the "special angles" on the unit circle (multiples of 30° and 45°). We could choose any point on the terminal side of each angle and the same ratios would result because the triangles used to calculate the ratios are similar.



$$\sin\alpha = \frac{y}{r} = \frac{y'}{r'} \qquad \cos\alpha = \frac{x}{r} = \frac{x'}{r'} \qquad \tan\alpha = \frac{y}{x} = \frac{y'}{x'}$$

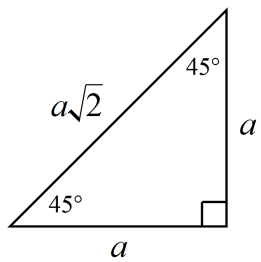
Since $r = 1$ for any point on the unit circle, points on the unit circle are convenient to use for calculating trigonometry ratios.

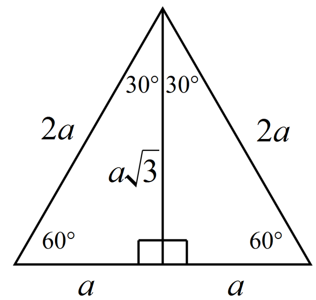
On the Unit Circle:

$$\sin\alpha = y \qquad \cos\alpha = x \qquad \tan\alpha = \frac{y}{x}$$

$$\csc\alpha = \frac{1}{y} \qquad \sec\alpha = \frac{1}{x} \qquad \cot\alpha = \frac{x}{y}$$

We can find the use the ratios that exist in special right triangles to calculate the coordinates of points on the unit circle.



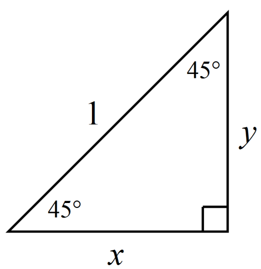


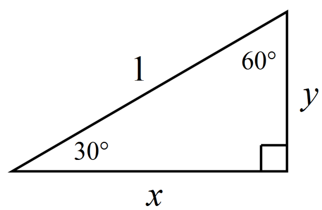
hypotenuse = leg $\cdot \sqrt{2}$

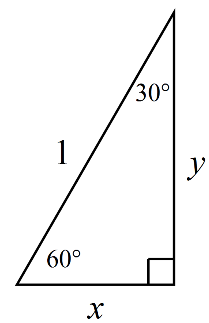
hypotenuse = 2 $\cdot$ short leg

long leg = short leg $\cdot \sqrt{3}$

Find the values of x and y in the triangles below:







The *signs* of the trigonometric functions depend on the quadrant in which the angle lies and the corresponding signs of x and y (remember r is always positive).

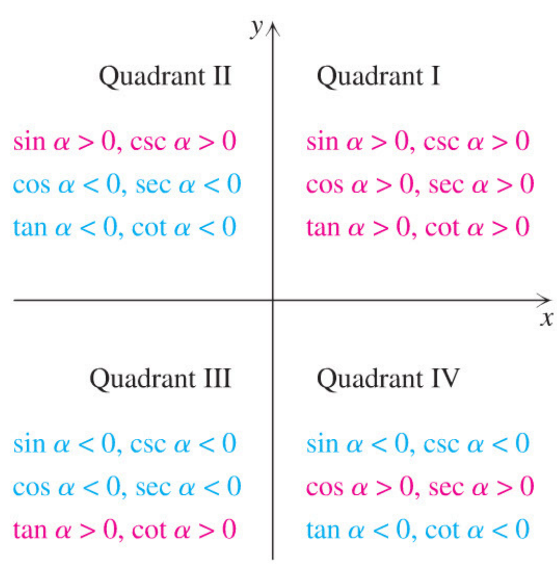
A good mnemonic to remember which functions are positive in each quadrant is "**A**ll **S**tudents **T**ake **C**alculus":

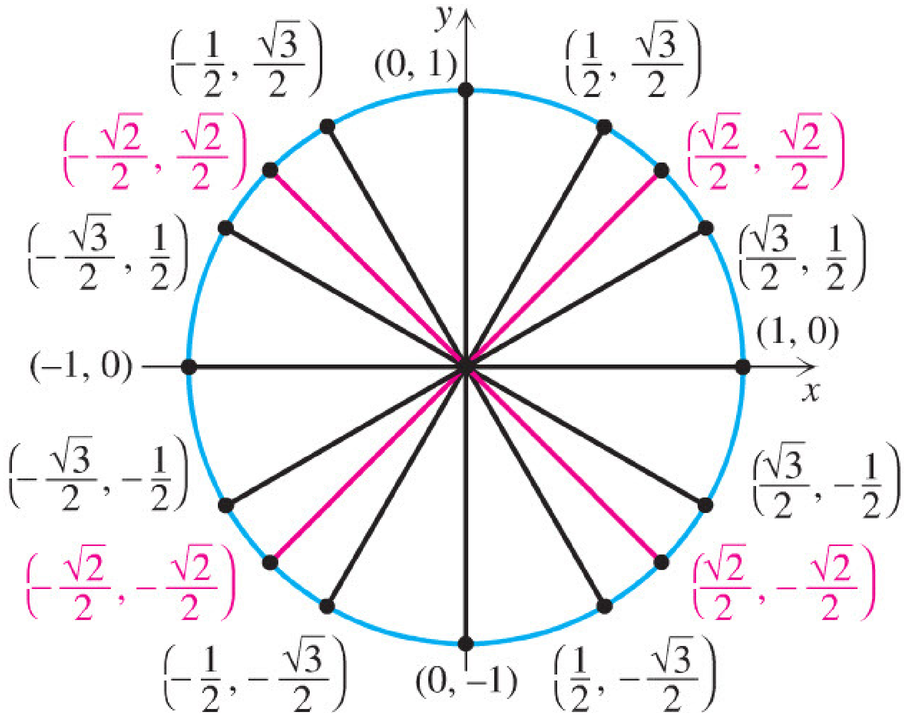
Quadrant I: All of them are positive

Quadrant II: sin and csc are positive

Quadrant III: tan and cot are positive

Quadrant IV: cos and sec are positive





Examples: Find the exact values of the following:

1. $\sin 0°$
2. $\cos \pi$
3. $\tan(-\pi/2)$
4. $\csc(-270°)$
5. $\sin(\pi/4)$
6. $\cos(-225°)$
7. $\cot(5\pi/4)$
8. $\sec 315°$
9. $\sin 30°$
10. $\cos(7\pi/6)$
11. $\tan(-\pi/3)$
12. $\csc 150°$
13. $\cot(-240°)$
14. $\sec(-\pi/6)$
15. $\cos(5\pi/3)$
16. $\tan(-150°)$

Right Triangle Trigonometry

A solution to the equation $\sin\alpha = \frac{1}{2}$ is an angle whose sine is $\frac{1}{2}$. Because $\sin 30° = \frac{1}{2}$ and $\sin 150° = \frac{1}{2}$, α could be 30° or 150°. Since any angle with the same terminal side as 30° or 150° is also a solution, there are infinitely many solutions. Since right triangles have only acute angles, we are only interested in the acute solutions in this section.

Examples:
Find the angle α that satisfies each equation where $0° \le \alpha \le 90°$.

a. $\sin\alpha = \sqrt{3}/2$ b. $\cos\alpha = 1$ c. $\tan\alpha = 1$

Inverse Sine, Cosine, and Tangent Functions
To calculate the size of angles with a given sine, cosine, or tangent, we use the inverse trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$, also known as arcsine (arcsin), arccosine (arccos), and arctangent (arctan).

★ **The -1 in $\sin^{-1} x$ does not indicate a reciprocal. $\sin^{-1} x \ne \frac{1}{\sin x}$. The -1 indicates an inverse function. $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ are angles!**

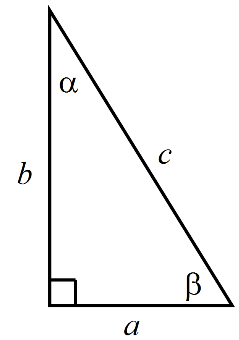
Because there are infinitely many angles that have a given sine, cosine, or tangent, we define the inverse functions precisely by restricting their domains:

- The inverse sine of x ($\sin^{-1} x$ or $\arcsin x$) is the angle between $-90°$ and $90°$ whose sine is x.
  - If $\sin\alpha = x$, and $-90° \le \alpha \le 90°$, then $\alpha = \sin^{-1} x$.
- The inverse cosine of x ($\cos^{-1} x$ or $\arccos x$) is the angle between $0°$ and $180°$ whose cosine is x.
  - If $\cos\alpha = x$, and $0° \le \alpha \le 180°$, then $\alpha = \cos^{-1} x$.
- The inverse tangent of x ($\tan^{-1} x$ or $\arctan x$) is the angle between $-90°$ and $90°$ whose tangent is x.
  - If $\tan\alpha = x$, and $-90° < \alpha < 90°$, then $\alpha = \tan^{-1} x$.

Examples:
Evaluate each expression. Give the result in degrees. Where necessary, round to the nearest tenth.

a. $\cos^{-1}\left(\sqrt{2}/2\right)$ b. $\arcsin\left(\sqrt{3}/2\right)$ c. $\tan^{-1}(6.1)$

Solving Right Triangles



Finding all the missing angle measures and side lengths of a triangle is called "solving a triangle". In a right triangle, we usually name the acute angles α and β (beta) and the lengths of the sides opposite those angles a and b, respectively. The 90° angle is γ (gamma) and the length of the side opposite the right angle (the hypotenuse) is c.

- If you know the lengths of two of the sides, use the Pythagorean Theorem to find the length of the third side.
- If you know the measure of one of the acute angles, use the fact that the angles in a triangle add to 180° to find the measure of the other angle.
- If you know the measure of one angle and the length of one side, use sin, cos, or tan to figure out the lengths of the other sides.
- If you know the lengths of the sides and need to figure out the angle measures, use inverse functions ($\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$).

Examples:

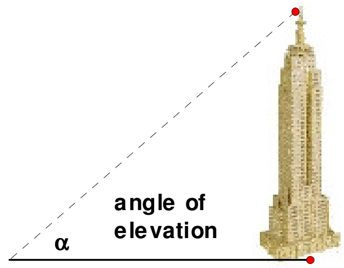
Solve the right triangle in which $\alpha = 60°$ and $c = 2$.

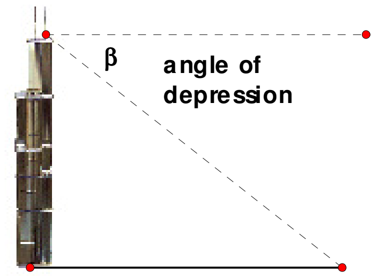
Solve the right triangle in which $a = 2$ and $b = 5$.

Solve the right triangle in which $\beta = 20°$ and $b = 15$.

Solve the right triangle in which $a = 5$ and $c = 13$.

Using trigonometry, we can find the size of an object without actually measuring the object. Two common terms used in this regard are **angle of elevation** and **angle of depression**.





Examples:
The angle of elevation of the top of a cell phone tower is 38.2° at a distance of 344 feet from the tower. What is the height of the tower?

At one location, the angle of elevation of the top of an antenna is 44.2°. At a point that is 100 feet closer to the antenna, the angle of elevation is 63.1°. What is the height of the antenna?

The Fundamental Identity and Reference Angles

The Fundamental Identity

The fundamental identity of trigonometry involves the squares of the sine and cosine function. We write $(\cos\alpha)^2$ as $\cos^2\alpha$ and $(\sin\alpha)^2$ as $\sin^2\alpha$. Remember that by definition, $\sin\alpha = y/r$, $\cos\alpha = x/r$, and $x^2 + y^2 = r^2$.

$$\sin^2\alpha + \cos^2\alpha = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1$$

The Fundamental Identity of Trigonometry: If α is any angle or real number, then $\sin^2\alpha + \cos^2\alpha = 1$.

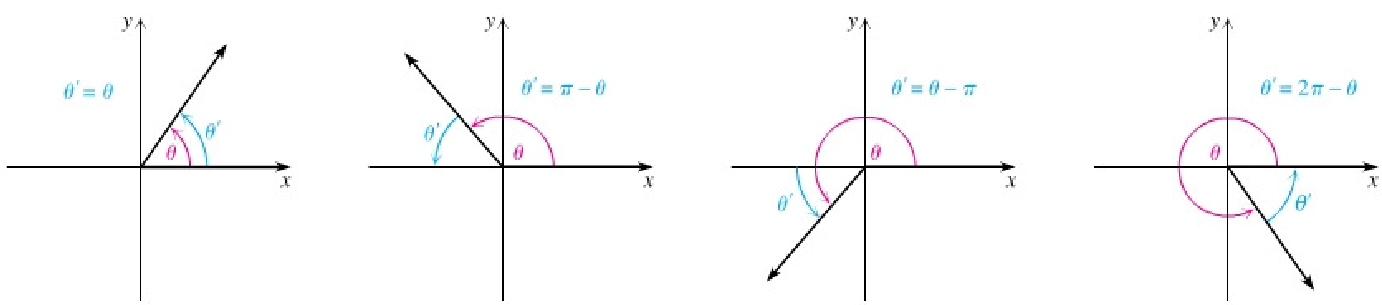
If we know the sine or cosine of an angle, then we can use the fundamental identity to find the value of the other function of the angle. (Note: you can also figure this out by drawing a triangle and using the Pythagorean Theorem).

Example: Find $\sin\alpha$ given that $\cos\alpha = 1/4$ and α is in Quadrant I.

Example: Find $\cos\alpha$ given that $\sin\alpha = -\sqrt{5}/3$ and α is in Quadrant III.

Reference Angles: When you look at the unit circle, notice that there is a pattern to the coordinates. If you look at all the angles that are 30° away from the *x*-axis (30°, 150°, 210°, 330°), the *x*-coordinate (cosine) is $\pm\sqrt{3}/2$ and the *y*-coordinate (sine) is $\pm 1/2$.

Definition: Reference Angle: If θ is a nonquadrantal angle (not on an axis) in standard position, then the reference angle θ' (read "theta prime") formed by the terminal side of θ and the positive or negative *x*-axis.



Examples: For each given angle θ, sketch the reference angle θ' and give the measure of θ' in both radians and degrees.

$\theta = 120°$ $\qquad$ $\theta = 7\pi/6$ $\qquad$ $\theta = 690°$ $\qquad$ $\theta = -7\pi/4$

Evaluating Trigonometric Functions Using Reference Angles: For an angle θ in standard position that is not a quadrantal angle:

$$\sin\theta = \pm\sin\theta', \qquad \cos\theta = \pm\cos\theta', \qquad \tan\theta = \pm\tan\theta',$$

$$\csc\theta = \pm\csc\theta', \qquad \sec\theta = \pm\sec\theta', \qquad \cot\theta = \pm\cot\theta'$$

where θ' is the reference angle for θ and the sign is determined by the quadrant in which θ lies.

Examples: Find the sine and cosine for each angle using reference angles.

$\theta = 120°$ $\qquad$ $\theta = 7\pi/6$ $\qquad$ $\theta = 690°$ $\qquad$ $\theta = -7\pi/4$

Modeling with the Sine Function

The trigonometric functions can be used to model periodic phenomena.

Examples: Demand for a seasonal product can be modeled by the function $d = 200\sin\dfrac{\pi(t-3)}{6} + 300,$ where d is the number of units sold in month t. Find the demand in March ($t = 3$) and June ($t = 6$).